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Proportionality and the power of unequal parties

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In this paper we introduce the concept of an overall power function that is meant to combine two sources of a party's power in a parliament. The first source is based on the possibilities for the party to be part of a majority coalition and it is typically modeled using a cooperative simple game. The second source takes into account parties' asymmetries outside the cooperative game and it is displayed by a vector of exogenously given weights. We adopt a normative point of view and provide an axiomatic characterization of a specific overall power function, in which the weights enter in a proportional fashion.

Key words external weights, overall power, proportionality, simple games, solution

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1 Introduction

The analysis of parliamentary election results is arguably one of the most prominent applications of cooperative game theory. In the present paper we tackle the following question: How should we measure *power* or *influence* of a party within the parliamentary system or, in particular, within a government. The latter can often be observed by considering how responsibilities (e.g. ministries) are distributed in the cabinet. Such a distribution frequently reflects the number of seats of the government parties, but disguises the fact that some parties might have more options to be part of some government.

More precisely, in the published literature one finds two approaches to measure differences in parties' power, each one ignoring the other. The first approach simply attaches to each party a number that reflects its "size" or "weight." This can be, for example, the number of seats in the parliament, the total number of voters having voted for that party or, where parties represent countries, the number of inhabitants or economic indices. Here, it is typically not the absolute value but the ratio of any such two numbers that has a meaningful interpretation.

The second approach departs from the institutional possibilities to form majorities in a parliament. Consequently, a (cooperative) *simple game* merely collects all coalitions of

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parties that might win an election within the parliament; that is, it collects all *winning coalitions*. Put differently, the data of a simple game ignore any notion of "size" of a party and solely reflect the possibilities to form governments. Within the framework of game theory, so called power indices, such as the Shapley–Shubik index or the Banzhaf index, are used to assess a party's strength or power within the simple game. In the parliament example, the Shapley–Shubik index of a simple game can readily be interpreted as parties' expected influences to decide an election with their vote and, therefore, serves in this sense as a measure of power.

One might ask whether either of the two methods to describe power alone can capture the relevant aspects. Obviously, this question is of a rhetorical nature. On the one hand, it is often observed that a party's bargaining position depends on its options to be part of the government. Therefore, the game and, hence, the power index, is of relevance. On the other hand, when it comes to negotiating over offices within a government, the size of the party clearly plays a non-negligible role. As a striking example, consider the German Bundestag between 1961 and 1980. There were two "large" parties (Christian Democratic Union/Christian Social Union and Social Democratic Party) and one "small" party (Free Democratic Party) in the parliament. Neither party had an absolute majority of seats. Hence, a winning coalition has to include at least two parties, rendering the simple game symmetric. As a power index solely rests on the data of the simple game, it therefore has to assign equal power to all three parties. However, the actual distribution of offices was far from equal whenever a government coalition of a large and the small party was formed. It was rather the relative size of a party that played a role in cabinet negotiations. In particular, power indices do not conform with ideas of proportionality with respect to asymmetries outside the game (see e.g. Snyder, Ting, and Ansolabehere (2005) and references therein for a debate on power indices).

Apart from the specific example of parties in a parliament, one may think of other scenarios that fit into our framework in the sense that the winning coalitions of the simple game do not reflect the size of the players. For example, bicameral elections such as the Senate and the House of Representatives in the US legislative or the German federal system with "Bundestag" (parliament) and "Bundesrat" (assembly of the federal governments) yield simple games that do not reflect sizes in the above sense. Even more striking is the situation with the threefold voting system in the European Union. The fact that a proposal has to pass three criteria (majority of population, seats and countries) can place considerable attention on smaller countries, as their votes can be crucial. More generally, one may also use our approach to assess the influence of countries in a political union. The decision structures define a simple game, but economic differences should also play a role. A further application is for local regions seeking more independence from a state.[1] Within the state, differences in natural resource endowments might be neglected, but these differences have to be taken into account when regions are organized as independent members of a (political) union. Therefore, our approach might help to determine how resources should be shared in such alliances. Finally, the decision system in the UN Security Council with its veto power given to the five permanent members allots disproportionately

[1] We thank an anonymous referee for suggesting this interesting line of interpretation.

high influence to these states that cannot be seen in any notion of size of a country. In principle, our setup fits for those situations, in which there is a definition of size of a player and in which there is a voting system that identifies certain coalitions of players as winning.

In the present paper, we use the following language to distinguish between the two measures of power. The simple game will be the basic ingredient and, therefore, we term the result of a power index as an *internal power* (distribution), as it is inherent in the game. By *external weights* (distribution) we mean the quantifications of party's size, where "external" indicates that this is not part of the description of the simple game. Phrased differently, external weights are independent of the current voting system, while internal power does not take into account parties' sizes. The goal of the present paper is to bring together the two concepts in order to arrive at a sound notion of *overall power* of a party.

More precisely, we shall study the overall power $F_i(\alpha, S, v, \varphi)$ of a party i within a coalition S when external weights are represented by the vector α and φ is the power index to measure internal power in the underlying game v.[2] We call such a function F an *overall power function* or, because it is a composite of two methods, a composite solution. In the present paper, we adopt a normative point of view and present a list of four axioms for overall power functions that should be satisfied, when combining external weights and internal power. Our theorem (Theorem 1) is that these four axioms are sufficient to uniquely characterize the specific composite solution, in which external weights enter proportionally. In other words, if those axioms should be met, then there is only one way to define overall power.

The idea of expressing differences in players' external characteristics through (strictly positive) weights goes back to Shapley (1953a), where it was used to characterize a particular parametrized solution (the weighted Shapley value (see Shapley 1953b; Owen 1968; Kalai and Samet 1987; Haeringer 2006; see also Radzik, Nowak, and Driessen (1997) for a discussion on weighted Banzhaf values). A *parametrized solution* assigns a share for each player to each pair (α, v). In contrast to the above papers, we take solution concepts φ defined on the set of all (simple) games rather than on the collection of all pairs (α, v). Hence, we let the players play the game as if there were no external asymmetries among them and, after that, we view external weights as having a redistributive role in an overall power function. This will be important when discussing the axioms we use for the characterization of our specific power function. Clearly, if we fix the solution concept φ and take only the grand coalition into account, then a parametrized solution (on simple games) can be seen as an overall power function as well.

The paper is organized as follows. Section 2 contains basic notions and definitions, such as the one of an overall power function. Moreover, we present and discuss four axioms for such functions. In Section 3 we show that these four axioms, being independent, characterize a specific overall power function. Section 4 closes with some final remarks.

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[2] In fact, instead of focusing on power indices, we cover a larger class of cooperative solution concepts.

2 Overall power functions and axioms

Let N be a finite *set of players* (e.g. parties or countries), which we will keep fixed throughout the paper. A (*cooperative*) *simple game with transferable utility* (a simple TU-game) is a pair (N, v), where $v : 2^N \to \{0, 1\}$ is called a *characteristic function* and satisfies $v(\emptyset) = 0$. We refer to a coalition $S \subseteq N$ with $v(S) = 1$ as a *winning coalition.* A winning coalition S is *minimal winning*, if neither of its strict subcoalitions is winning; that is, $v(T) = 0$ for all $T \subsetneq S$. In what follows, we will identify a simple game (N, v) with its characteristic function v.

For $S \in 2^N$ define the *restricted game with respect to* S, denoted (N, v_S), by $v_S(T) = v(S \cap T)$ for all $T \in 2^N$. Note that v_S is an N-player game (possibly with $v_S(N) = v(S) = 0$). The set of all simple games on the player set N will be denoted by $\mathcal{G}$. A game $v \in \mathcal{G}$ is *monotonic* if $v(S) = 1$ implies $v(T) = 1$ for all $T \supseteq S$. The set of all monotonic simple games on the player set N is denoted by $\mathcal{G}^m$. Clearly, if a game v is in the set $\mathcal{G}^m$, then so are any of its restricted games.[3]

A *solution* (for a simple TU-game) is a mapping $\varphi : \mathcal{G} \to \mathbb{R}^N$ taking each $v \in \mathcal{G}$ to a single vector in $\mathbb{R}^N$; that is, it assigns a real number $\varphi_i(v)$ to each player $i \in N$. Later, we may interpret the number $\varphi_i(v)$ as player i's power in v and as we only rely on the data of the game, we term this player i's *internal power*. The set of all solutions on $\mathcal{G}$ will be denoted by $\mathcal{S}$. A solution $\varphi \in \mathcal{S}$ is *positive* if $\varphi_i(v) \geq 0$ $(i \in N)$ holds for all $v \in \mathcal{G}^m$. The set of all positive solutions on $\mathcal{G}$ will be denoted by $\mathcal{S}^+$.[4]

Overall power functions, as defined next, are designed to incorporate external weights as well as internal power. Asymmetries outside the game shall be captured by a weight vector $\alpha \in \mathbb{R}^N_{++}$, which is to be combined with a solution φ. Therefore, an overall power function does not only reflect players' opportunities within a cooperative simple game, but also respects differences in external weights. Consequently, the distinction between *internal* (power) and *external* (weights) is based on the fact that any solution $\varphi \in \mathcal{S}$ reflects parties' power when only the cooperative simple game v is taken into account and in that sense it describes their power in v, while the weights are not used when defining the game and describe asymmetries among the players that cannot be captured by v.

Assuming that the (winning) coalition S forms, the question is, how influential are the members in S? The vector α describes external weights, v (respectively, v_S) the voting system and φ assesses internal power within the simple game. An overall power function F now assigns a number to each party i in such a coalition S, depending on v and φ.

Therefore, the domain of F is a subset $\mathcal{D}^0$ of $\mathcal{D} := \mathbb{R}^N_{++} \times 2^N \times \mathcal{G} \times \mathcal{S}$. Formally, an *overall power function* $F : \mathcal{D}^0 \longrightarrow \mathbb{R}^N$ on $\mathcal{D}^0$ assigns for each player a real number to each tuple $(\alpha, S, v, \varphi) \in \mathcal{D}^0$ consisting of a strictly positive weight vector, a coalition, a simple game and a solution. We shall interpret the function F as follows. Suppose that the game v represents the possibilities to form winning coalitions. The vector α reflects asymmetries

[3] Note that any majority voting game is monotonic; that is, a coalition is winning if its total number of seats exceeds the majority quota.

[4] Monotonicity guarantees that adding a new party can never render a winning coalition into a losing one. Positivity of a solution (power index) simply assures that in such cases no party is assigned a negative power index.

outside v and φ is the solution that measures players' (internal) power in v. Provided that a coalition $S \in 2^N$ has formed, we view the real number $F_i(\alpha, S, v, \varphi)$ as "player i's overall power" in S. Put another way, F merges a player's external weight and his or her internal power to arrive at a notion of his or her overall power (within a coalition S). The reason why an overall power function F is defined on some subset of $\mathcal{D}$ is that it allows us to focus on specific simple games or particular solution concepts. As a matter of fact, in the next section we restrict our attention to monotonic simple games and positive solutions.

Observe that we do not assume that external weights, α, are connected to the game v. In particular, α need not be a representation in the sense that a coalition S is winning ($v(S) = 1$) if and only if its total external weight exceeds a certain quota.

The main objective now is to find a "correct way" to combine players' internal power and external weights within an overall power function. For this, we first define a list of plausible properties that such a function should meet. In the next section, we show that there is one and only one overall power function that satisfies all properties of that list. The way in which two measures of power are brought together is uniquely determined by the following four axioms.

All axioms rely on the following situation. Suppose coalition S has formed and an overall power distribution (among all players in N) shall be determined by an overall power function.

The first axiom, Ignorance of Outsiders, reflects the natural requirement that all "outsiders," that is, players who are not members of S, shall not have power when S is formed.

Ignorance of Outsiders (IO): For all $(\alpha, S, v, \varphi) \in \mathcal{D}^0$ and all $i \in N \setminus S$,

$$F_i(\alpha, S, v, \varphi) = 0.$$

The remaining three axioms concentrate on the distribution of overall power within coalition S. Recall that the vector $\varphi(v)$ describes (internal) power within the grand coalition N (in the unrestricted game $v = v_N$). Consequently, when we discuss power distributions within a smaller coalition S, the restricted game v_S is the relevant part of v. Hence, we think of $\varphi(v_S)$ as the (internal) power distribution within S that is proposed by solution φ.[5]

Our second axiom, Sign Inheritance, requires that the incorporation of external weights can neither create positive overall power, when the internal power is zero, nor can it destroy a positive internal power. Hence, this axiom rules out extreme cases in which only players' external characteristics (captured by α) determine their overall power in a coalition. As an example, any power index assigns zero internal power to parties that are never members of minimal winning coalitions; that is, minimal governments. Such dummy parties can never alter a decision. The Sign Inheritance axiom guarantees that dummy parties will be

[5] For instance, while $\varphi(v)$ reflects the (internal) distribution of power in the parliament, $\varphi(v_S)$ does so within the government S.

assigned zero overall power. Similarly, parties with positive internal power, that is, which have an influence on some decisions, shall retain positive overall power. Hence, F_i inherits its sign from φ_i.

Sign Inheritance (SI): For all $(\alpha, S, v, \varphi) \in \mathcal{D}^0$ and all $i \in S$,

$$\varphi_i(v_S) > 0 \Rightarrow F_i(\alpha, S, v, \varphi) > 0 \text{ and } \varphi_i(v_S) = 0 \Rightarrow F_i(\alpha, S, v, \varphi) = 0.$$

Our third axiom, Power Redistribution, emphasizes the redistributive role of external weights. It requires that the total overall power that is distributed within coalition S amounts to the total internal power that the φ distributes within S. In particular, the total (internal) power that a power index assigns to a winning coalition (government) S is always equal to 1. Now, independent of government parties' external weights, it appears plausible that the total overall power likewise should be 1.

Power Redistribution (PR): For all $(\alpha, S, v, \varphi) \in \mathcal{D}^0$,

$$\sum_{k \in S} F_k(\alpha, S, v, \varphi) = \sum_{k \in S} \varphi_k(v_S).$$

To motivate our final axiom, Constant Transformation Rates per Weight, we go through a couple of thought experiments. First, suppose that there are two parties i, j in S having identical possibilities to be part of a government, which are, therefore, allocated the same internal power by φ; that is, $\varphi_i(v_S) = \varphi_j(v_S)$. In contrast, if player i's weight is twice as high as player j's weight,[6] then we consider it reasonable to assign twice as much overall power to party i than to j. If internal power is equal, the axiom stipulates that the ratio of overall power matches the ratio of players' weights. Second, the situation should be similar when two parties have the same external weight (e.g. the same number of seats). Then, if one party is assigned twice as much internal power (by φ), then it should have twice as much overall power as well.

The ratio F_i/φ_i describes the rate at which player i's internal power is transformed to arrive at his or her overall power value. To make it more precise, the higher this ratio, the more his or her internal power is taken into account (or weighted) or, in other words, the "stronger" this player is. However, because asymmetries (here in "strength") are captured by α, the following axiom requires that external weights reflect such transformation rates.[7]

Constant Transformation Rates per Weight (CTW): For all $(\alpha, S, v, \varphi) \in \mathcal{D}^0$ and all $i, j \in S$ with $\varphi_i(v_S) > 0$ and $\varphi_j(v_S) > 0$,

$$\frac{F_i(\alpha, S, v, \varphi)}{\varphi_i(v_S)} \Big/ \alpha_i = \frac{F_j(\alpha, S, v, \varphi)}{\varphi_j(v_S)} \Big/ \alpha_j.$$

6 There are even more drastic examples, where the number of seats of a party stands in stark contrast to its chances of being in a government (see e.g. the current German Bundestag).

7 In this motivation for the axiom, we use the term "strength" not as a well-defined technical term, but rather to illustrate that there might be asymmetries between transformation rates.

Using the above language, the CTW axiom requires that the ratio of two players' transformation rate is given by the ratio of their external weights; that is, equally "strong players" have the same transformation rate.

3 A characterization result

In this section we define a specific overall power function Φ and demonstrate that it is the only one that satisfies the four axioms from the previous section. It is frequently observed that external weights (e.g. seats or population) are used in relative terms. For example, when it comes to distributing offices in a government, the ratios of numbers of parties' seats serves as a focal point for the ratios of offices. Consequently, we define a "proportional version" of an overall power function, denoted Φ, in which external weights reweigh internal power proportionally, as follows:

As the domain for Φ we take $\bar{\mathcal{D}} := \mathbb{R}^N_{++} \times 2^N \times \mathcal{G}^m \times \mathcal{S}^+ \subset \mathcal{D}$, which restricts attention to monotonic simple games and positive solutions.[8] In particular, $\bar{\mathcal{D}}$ covers all power indices as possible solutions. Next, define $\Phi : \bar{\mathcal{D}} \longrightarrow \mathbb{R}^N$ by[9]

$$\Phi_i(\alpha, S, v, \varphi) = \begin{cases} \dfrac{\alpha_i \varphi_i(v_S)}{\sum_{k\in S} \alpha_k \varphi_k(v_S)} \cdot \sum\limits_{k \in S} \varphi_k(v_S), & i \in S, \\ 0, & i \notin S \end{cases} \qquad (i \in N). \tag{1}$$

Therefore, within a coalition S, $\sigma := \sum_{k\in S} \varphi_k(v_S)$ is redistributed among the players in S.[10] Without external weights, player i's $(i \in S)$ share of σ was $\varphi_i(v_S)/\sum_{k\in S}\varphi_k(v_S)$. This share is now multiplied (weighted) with his or her external weight, α_i. Normalization, that is, the sum of shares equals 1, yields the form in (1). Note also that Φ is homogeneous of degree zero with regard to the weights vector, which means that only relative weights among players matter.

Interestingly, one implication of the form in (1) is that within minimal governments, power is shared proportionally to external weights. More precisely, if S is minimal winning, then in the restricted game v_S all parties in S are symmetric as the whole coalition S is necessary to obtain power. Therefore, any power index (precisely, any symmetric solution) assigns equal internal power. This also reveals why power indices are clearly not sufficient to describe power distributions within a government. Finally, using Φ, party i's overall power in the minimal winning coalition S is $\alpha_i / \sum_{j\in S}\alpha_j$. It is evident that it is mainly the CTW axiom that ensures this feature of the function Φ.

Theorem 1 *An overall power function $F : \bar{\mathcal{D}} \longrightarrow \mathbb{R}^N$ satisfies IO, SI, PR and CTW if and only if $F = \Phi$.*

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8 Recall that for positive solutions on monotonic games, $\varphi_i(v) \in R^N_+$ and, therefore, we may interpret this value as i's power.

9 We use the convention that $\frac{0}{0} = 0$ (e.g. for the case that v_S is the zero game).

10 Note that if S is winning and φ is efficient, meaning $\varphi(v_S)$ always distributes the total power of S, then $\sigma = v_S(S) = 1$.

PROOF: The proof proceeds in two steps. In Step 1 we show that the overall power function Φ satisfies the four axioms on its domain. Then, in Step 2, we prove that an overall power function satisfying the four axioms takes the form of Φ.

Step 1: Φ satisfies the four axioms on $\bar{\mathcal{D}}$

By construction, Φ clearly satisfies IO. On the domain $\bar{\mathcal{D}}$, Φ always assigns non-negative values, because any considered game v is monotonic and φ is positive. From the construction and recalling that weights are strictly positive, is is readily seen that $\Phi_i(\alpha, S, v, \varphi) > 0$ holds if and only if $\varphi_i(v_S) > 0$ is true. Hence, SI is satisfied.

Next, for all $(\alpha, S, v, \varphi) \in \bar{\mathcal{D}}$ we have

$$\sum_{i \in S} \Phi_i(\alpha, S, v, \varphi) = \frac{\sum_{i \in S} \alpha_i \varphi_i(v_S)}{\sum_{k \in S} \alpha_k \varphi_k(v_S)} \sum_{k \in S} \varphi_k(v_S) = \sum_{k \in S} \varphi_k(v_S),$$

which shows that the PR axiom is also fulfilled.

Finally, we establish CTW. Take $(\alpha, S, v, \varphi) \in \bar{\mathcal{D}}$, $i, j \in S$ with $\varphi_i(v_S) > 0$ and $\varphi_j(v) > 0$. Then we obtain

$$\frac{\frac{\Phi_i(\alpha, S, v, \varphi)}{\varphi_i(v_S)}}{\alpha_i} = \frac{\alpha_i \varphi_i(v_S) \sum_{k \in S} \varphi_k(v_S)}{\alpha_i \varphi_i(v_S) \sum_{k \in S} \alpha_k \varphi_k(v_S)} = \frac{\sum_{k \in S} \varphi_k(v_S)}{\sum_{k \in S} \alpha_k \varphi_k(v_S)}$$

$$= \frac{\alpha_j \varphi_j(v_S) \sum_{k \in S} \varphi_k(v_S)}{\alpha_j \varphi_j(v_S) \sum_{k \in S} \alpha_k \varphi_k(v_S)} = \frac{\frac{\Phi_j(\alpha, S, v, \varphi)}{\varphi_j(v_S)}}{\alpha_j},$$

which is the condition in CTW.

Step 2: Uniqueness

Let $F : \bar{\mathcal{D}} \longrightarrow \mathbb{R}^N$ satisfy the above four axioms. Take $i \in N$ and $(\alpha, S, v, \varphi) \in \bar{\mathcal{D}}$, and consider the following four cases:

Case 1 ($i \in N \setminus S$): Then, by IO, $F_i(\alpha, S, v, \varphi) = 0 = \Phi_i(\alpha, S, v, \varphi)$.

Case 2 ($S = \{i\}$): By PR, $F_i(\alpha, S, v, \varphi) = \varphi_i(v_S)$, which is, by definition, equal to $\Phi_i(\alpha, S, v, \varphi)$.

Case 3 ($i \in S$, $|S| \geq 2$ and $\varphi_i(v_S) = 0$): By SI, $F_i(\alpha, S, v, \varphi) = 0$, which again, by definition, is equal to $\Phi_i(\alpha, S, v, \varphi)$.

Case 4 ($i \in S$, $|S| \geq 2$ and $\varphi_i(v_S) > 0$): We consider the following two subcases.

Subcase 4.1 ($\varphi_k(v_S) = 0$ for all $k \in S \setminus \{i\}$): We have

$$F_i(\alpha, S, v, \varphi) = \sum_{k \in S} \varphi_k(v_S) - \sum_{k \in S \setminus \{i\}} F_k(\alpha, S, v, \varphi) = \varphi_i(v_S) = \Phi_i(\alpha, S, v, \varphi),$$

where the first equation follows from PR and the second from SI.

Subcase 4.2 ($\varphi_k(v_S) > 0$ for some $k \in S \setminus \{i\}$): Let $S' = \{l \in S \setminus \{i\} : \varphi_l(v_S) > 0\}$. Then,

$$
\begin{aligned}
F_i(\alpha, S, v, \varphi) &= \sum_{k \in S} \varphi_k(v_S) - \sum_{k \in S \setminus \{i\}} F_k(\alpha, S, v, \varphi) \\
&= \sum_{k \in S} \varphi_k(v_S) - \sum_{k \in S'} F_k(\alpha, S, v, \varphi) - \sum_{k \in (S \setminus \{i\}) \setminus S'} F_k(\alpha, S, v, \varphi) \\
&= \sum_{k \in S} \varphi_k(v_S) - \sum_{k \in S'} F_k(\alpha, S, v, \varphi) \\
&= \sum_{k \in S} \varphi_k(v_S) - \sum_{k \in S'} \frac{F_i(\alpha, S, v, \varphi)\, \alpha_k \varphi_k(v_S)}{\alpha_i \varphi_i(v_S)} \\
&= \sum_{k \in S} \varphi_k(v_S) - \frac{F_i(\alpha, S, v, \varphi)}{\alpha_i \varphi_i(v_S)} \sum_{k \in S'} \alpha_k \varphi_k(v_S) \\
&= \sum_{k \in S} \varphi_k(v_S) - \frac{F_i(\alpha, S, v, \varphi)}{\alpha_i \varphi_i(v_S)} \sum_{k \in S \setminus \{i\}} \alpha_k \varphi_k(v_S),
\end{aligned}
\tag{2}
$$

where the first equation follows from PR, the third one from $\varphi \in \mathcal{S}^+$ and SI, the fourth one from CTW, and the last one from $\varphi_k(v_S) = 0$ for each $k \in (S \setminus \{i\}) \setminus S'$. Rearranging (2) leads to:

$$
\begin{aligned}
F_i(\alpha, S, v, \varphi) &= \frac{\sum_{k \in S} \varphi_k(v_S)}{\left(1 + \dfrac{\sum_{k \in S \setminus \{i\}} \alpha_k \varphi_k(v_S)}{\alpha_i \varphi_i(v_S)}\right)} \\
&= \frac{\alpha_i \varphi_i(v_S)}{\sum_{k \in S} \alpha_k \varphi_k(v_S)} \sum_{k \in S} \varphi_k(v_S) = \Phi_i(\alpha, S, v, \varphi).
\end{aligned}
$$

The four cases together establish $F = \Phi$. □

Let us finally show that the axioms used for the characterization of Φ are logically independent. That means neither axiom is implied by the remaining ones and is therefore necessary for the characterization. A standard way to show this is to find four overall power functions each of which satisfying all axioms but one.

Example 1 Define the overall power function $F^1 : \bar{\mathcal{D}} \longrightarrow \mathbb{R}^N$ by

$$F_i^1(\alpha, S, v, \varphi) = \begin{cases} \sum_{k \in S} \varphi_k(v_S), & \text{if } S = \{i, j\},\ j \neq i, \text{ and } \\ & \varphi_j(v_S) > \varphi_i(v_S) = 0, \\ 0, & \text{if } S = \{i, j\},\ j \neq i, \text{ and } \\ & \varphi_i(v_S) > \varphi_j(v_S) = 0, \\ \Phi_i(\alpha, S, v, \varphi), & \text{otherwise} \end{cases} \qquad (i \in N).$$

It is easy to check that, by its construction, F^1 satisfies PR and IO. It also satisfies CTW because this axiom only states a requirement for coalitions S, in which there are $i, j \in S$ with $\varphi_i(v_S) > 0$ and $\varphi_j(v_S) > 0$, and we have $F^1 = \Phi$ in such cases. However, F^1 violates SI because for $S = \{i, j\},\ j \neq i$, with $\varphi_j(v_S) > \varphi_i(v_S) = 0$ we have $F_i^1(\alpha, S, v, \varphi) = \varphi_j(v_S) > 0$.

Example 2 Next, consider $F^2 : \bar{\mathcal{D}} \rightarrow \mathbb{R}^N$ defined by

$$F_i^2(\alpha, S, v, \varphi) = \begin{cases} \frac{\alpha_i \varphi_i(v_S)}{\alpha(N)} \cdot \sum_{k \in S} \varphi_k(v_S), & \text{if } i \in S, \\ 0, & \text{otherwise} \end{cases} \qquad (i \in N),$$

where $\alpha(N) := \sum_{i \in N} \alpha_i$ denotes the total sum of external weights. Clearly, this solution satisfies SI and IO, and it is easy to check that it also satisfies CTW. However, F^2 violates PR, because

$$\sum_{i \in S} F_i^2(\alpha, S, v, \varphi) = \frac{\sum_{i \in S} \alpha_i \varphi_i(v_S)}{\alpha(N)} \cdot \sum_{k \in S} \varphi_k(v_S) \neq \sum_{i \in S} \varphi_k(v_S).$$

Example 3 Let $F^3 : \bar{\mathcal{D}} \rightarrow \mathbb{R}^N$ be defined by

$$F_i^3(\alpha, S, v, \varphi) = \begin{cases} \dfrac{\alpha_i \varphi_i(v_S)}{\sum_{k \in S} \alpha_k \varphi_k(v_S)} \cdot \sum\limits_{k \in S} \varphi_k(v_S), & \text{if } i \in S, \\ \dfrac{\alpha_i \varphi_i(v_S)}{\sum_{k \in N \setminus S} \alpha_k \varphi_k(v_S)} \cdot \sum\limits_{k \in N \setminus S} \varphi_k(v_S), & \text{otherwise} \end{cases} \qquad (i \in N).$$

As F^3 coincides with Φ for all members in S, inspection of Step 1 reveals that this solution satisfies all axioms but IO.

Example 4 Define $F^4 : \bar{\mathcal{D}} \rightarrow \mathbb{R}^N$ by

$$F_i^4(\alpha, S, v, \varphi) = \begin{cases} \dfrac{(\varphi_i(v_S))^{\alpha_i}}{\sum_{k \in S} (\varphi_k(v_S))^{\alpha_k}} \cdot \sum\limits_{k \in S} \varphi_k(v_S), & \text{if } i \in S, \\ 0, & \text{otherwise} \end{cases} \qquad (i \in N),$$

and notice that it does not satisfy CTW because, for $i, j \in S$ with $\varphi_i(v_S) > 0$ and $\varphi_j(v_S) > 0$,

$$\frac{\frac{F_i^4(\alpha, S, v, \varphi)}{\varphi_i(v_S)}}{\alpha_i} = \frac{(\varphi_i(v_S))^{\alpha_i}}{\alpha_i \varphi_i(v_S)} \cdot \frac{\sum_{k \in S} \varphi_k(v_S)}{\sum_{k \in S} (\varphi_k(v_S))^{\alpha_k}}$$

$$\neq \frac{(\varphi_j(v_S))^{\alpha_j}}{\alpha_j \varphi_j(v_S)} \cdot \frac{\sum_{k \in S} \varphi_k(v_S)}{\sum_{k \in S} (\varphi_k(v_S))^{\alpha_k}} = \frac{\frac{F_j^4(\alpha, S, v, \varphi)}{\varphi_j(v_S)}}{\alpha_j}.$$

It is easy to check that F^4 satisfies SI, PR and IO.

4 Concluding remarks

An overall power function measures players' power in a coalition when their internal power in a simple game is given by a cooperative solution concept, and a vector of positive weights describes asymmetries among the players outside the game. Traditionally, power indices, such as the Shapley–Shubik index, propose a distribution of power among the players in a coalition. However, a frequent criticism regarding power indices is that they ignore characteristics besides the data of the game (e.g. distribution of seats in a parliament). Overall power functions provide a tool to incorporate both external weights and the data of the game to arrive at an overall power distribution within a winning coalition. That means they provide an answer regarding how responsibilities can be distributed within a government. With the theorem from the previous section we have via the axioms justified a way to combine the two measurements by using external weights; they reweigh the internal power distribution.

In addition to the properties used for the characterization of Φ, this overall power function satisfies other desirable properties. For example, Φ inherits standard properties from the solution φ, such as efficiency or the null player property. Moreover, if φ is symmetric, then the ratio of overall powers of any two players who are symmetric in v matches the ratio of their external weights.

The overall power function Φ was first used in Dimitrov and Haake (2008) to develop and analyze a notion of stability of government coalitions.[11] In fact, for fixed α, v, φ the function $F_i(\alpha, \cdot, v, \varphi)$ can be seen as a representation of preferences over coalitions that player i is a member of. In other words, a party assesses the possible governments it can be part of according to how many offices it gets, measured by the overall power function Φ.

The works by Laruelle and Valenciano (2008) and Radzik, Nowak, and Driessen (1997) give rise to another line of reasoning for the CTW axiom. Suppose two parties have the same internal power, given by φ. Interpreting α as the size of the party measured in members (or

[11] In Dimitrov and Haake (2008), it is termed a "composite solution."

parliamentarians), the CTW axiom requires that any member of the two parties should be assigned the same (overall) power; that is, F_i/α_i is constant. In general, the overall power per member is a multiple of his or her party's internal power and the factor is common to all parliamentarians.

Finally, the reader may easily verify that if, on the one hand, we assume equal weights, then the overall power function Φ coincides with the solution φ.[12] On the other hand, suppose that internal differences among the players in the game are ignored by the solution concept, meaning that φ always distributes power equally among the members of a coalition.[13] Then, the ratio of overall powers of any two players is their ratio of external weights. Hence, Φ is essentially determined by the weights vector α.

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[12] More precisely, we have $\Phi_i(\alpha, S, v, \varphi) = \varphi_i(v_s)$ for all $(\alpha, S, v, \varphi) \in \bar{D}, i \in S$.

[13] For example, $\varphi_i(v_S) = \frac{v_S(S)}{|S|}$ for each $i \in S \subseteq N$.